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The critical Reynolds number of a laminar incompressible mixing layer from minimal composite theory

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According to parallel-flow theory based on the Orr–Sommerfeld equation, a mixing layer is unstable at all Reynolds numbers. However this is untenable from energy considerations, which demand that there exist a non-zero Reynolds number below which disturbances cannot extract net energy from the mean flow. It is shown here that a linear stability analysis of similarity solutions of the plane mixing layer, including the effects of flow non-parallelism using the minimal composite theory and the properties of adjoints, following Govindarajan & Narasimha (*Theor. Comput. Fluid Dyn.* vol. 19, 2005, p. 229) resolves the issue by yielding a non-zero critical Reynolds number for co-flowing streams of any velocity ratio. The critical Reynolds number for the total disturbance kinetic energy is found to be close to 30 for all velocity ratios in the range from zero to unity.

1. Introduction

The motivation for the present work arises from a well-known analysis of the stability of a mixing layer due to Betchov & Szewczyk (1963). This analysis, based on the Orr–Sommerfeld (OS) equation, showed that the marginal stability curve in the wavenumber (α)–Reynolds number (Re) plane approaches the origin $\alpha = 0$ as Re goes to 0, leading the authors to conclude "No minimum [i.e. critical] Reynolds number is found" for the flow. Mixing layers have been studied extensively since, and recent work, e.g. Huerre & Rossi (1998), Criminale, Jackson & Joslin (2003, p. 90) and Drazin & Reid (2004, pp. 199–201), do not report any analysis yielding a non-zero critical Reynolds number. This situation is intriguing, for several early studies of stability (e.g. Prandtl 1935, pp. 178–183; Lin 1955, pp. 31–32, 59+, going back to the two-dimensional analysis of 1907, pp. 43–71), show that (under certain reasonable conditions) a two-dimensional incompressible viscous flow must be stable at sufficiently low Reynolds numbers. The mixing layer being a very basic flow type, the absence of any definitive commentary on the above issue has led other studies of flows modelled on the mixing layer (e.g. Solomon, Holloway & Swinney 1993; Östekin, Cumbo & Liakopoulos 1999) to base their analyses on the assumption that the critical Reynolds number is zero.

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All existing analyses of mixing layer stability, from Esch (1957) to Balsa (1987), start with the OS equation, which is valid only for strictly parallel flow. However, the width of a laminar mixing layer scales as $x^{1/2}$ (where x is the streamwise distance), so the rate of change of thickness (and hence also any parameter measuring the degree of flow non-parallelism) becomes infinite in the limit of Re (equivalently x) going to 0. In other words, existing studies based on the parallel flow assumption yield a result which is incompatible with the assumption. Thus flow non-parallelism may be expected to play a crucial role in determining the stability characteristics in the limit. This was realized earlier; e.g. Bun & Criminale (1994) (after presenting an inviscid analysis) note that "with viscous effects, the basic flow should be treated as non-parallel", and Drazin & Reid (2004, p. 197) emphasize that "at small values of Reynolds number the parallel-flow assumption is of questionable validity". We show here that a consistent non-parallel flow theory yields a finite, non-zero critical Reynolds number.

The similarity solution of the laminar mixing layer is used as the base flow everywhere in the present analysis, as its use through what we have called minimal composite theory (Narasimha & Govindarajan 2000) is both appropriate and convenient in the present approach.

The rest of the paper is arranged as follows. In § 2 the similarity solution of the mean flow profile is presented briefly. The stability problem is posed and the method of solution outlined in § 3, essentially following the approach used for boundary layers by Govindarajan & Narasimha (1997, 2005). In § 4, the results of the stability analysis are presented and compared with earlier work, and in § 5 some error estimates are discussed.

2. The mean flow

The mean flow is governed by the Blasius similarity equation (Lock 1951; Schlichting & Gersten 2004; Monkewitz & Huerre 1982)

$$f''' + \tfrac{1}{2} f f'' = 0, \tag{2.1}$$

where

$$\left.\begin{aligned} f(\eta) &\equiv f_d(x_d, y_d)/U_\infty l_d(x_d), \qquad \eta \equiv y_d/l_d, \\ l_d(x_d) &\equiv (\nu x_d/U_\infty)^{1/2}, \qquad f' \equiv \mathrm{d}f/\mathrm{d}\eta, \end{aligned}\right\} \tag{2.2}$$

x_d, y_d are streamwise and normal coordinates and ν is the kinematic viscosity. With $U_{\pm\infty}$ being the free-stream velocity at $y_d \to \pm\infty$, the boundary conditions are

$$\left.\begin{aligned} &f'(+\infty) = 1, \qquad f'(-\infty) = (1 - \Lambda)/(1 + \Lambda), \\ &f'''(0) = 0, \qquad \text{where} \qquad \Lambda \equiv (U_\infty - U_{-\infty})/(U_\infty + U_{-\infty}). \end{aligned}\right\} \tag{2.3}$$

Solutions for $\Lambda = 1, 1/3, 1/7$ are compared with a tanh-profile in figure 1. We define the vorticity thickness of the layer as

$$\delta_\omega(x_d) \equiv \left(\frac{2\Lambda}{1+\Lambda}\right)\frac{l_d(x_d)}{f''_{max}}, \tag{2.4}$$

where f''_{max} is the maximum velocity gradient in the mixing layer.

3. The non-parallel stability problem

A brief outline of the minimal composite theory is described in what follows. Reviews of the method can be found in Narasimha & Govindarajan (2000, 2001), and Govindarajan & Narasimha (1997, 2005, referred to below as GN97 and GN05).

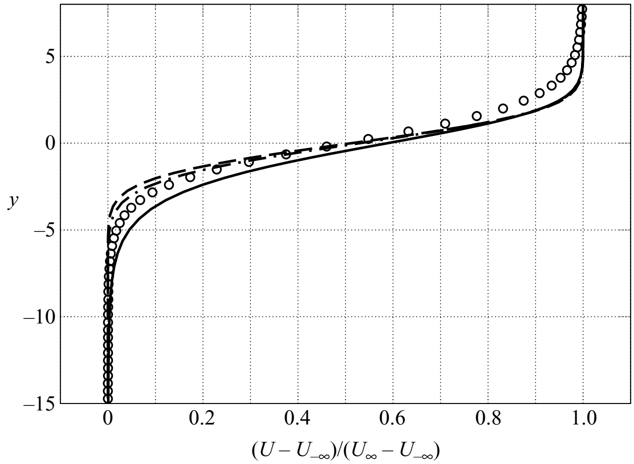

FIGURE 1. Streamwise velocity profiles from the solution of (2.1) for different velocity ratios. Solid line, similarity profile for $\Lambda = 1$; circles, $\frac{1}{2}\{1 + \tanh(2y/5)\}$; dash-dotted line, similarity profile for $\Lambda = 1/3$; dashed line, similarity profile for $\Lambda = 1/7$.

Each flow quantity, for example the stream function, is expressed as the sum of a mean Φ_d and a perturbation $\hat{\phi}_d$, where

$$\Phi_d(x_d, y_d) = \left(\frac{2\Lambda}{1+\Lambda}\right) U_\infty \delta_\omega(x_d) \Phi(y), \tag{3.1}$$

$$\hat{\phi}_d(x_d, y_d) = \left(\frac{2\Lambda}{1+\Lambda}\right) U_\infty \delta_\omega(x_d) \phi(x, y) \exp\left[\mathrm{i}\left(\int \alpha(x)\,\mathrm{d}x - \omega t\right)\right]. \tag{3.2}$$

Note the scaling with δ_ω and the velocity differential across the mixing layer,

$$\frac{U_\infty - U_{-\infty}}{U_\infty} = \frac{2\Lambda}{1+\Lambda}.$$

Furthermore, $y = y_d/\delta_\omega$ (=0 at the inflection point), and $\mathrm{d}x = \mathrm{d}x_d/\delta_\omega$. Inserting these into the linearized Navier–Stokes equations for two-dimensional incompressible flow written in terms of the stream function, and retaining all terms nominally up to $O(Re^{-1})$, the non-parallel stability equation can be written as

$$\mathcal{N}\{\phi\} = 0, \qquad \text{with boundary conditions} \quad \phi, \mathrm{D}\phi \to 0 \quad \text{at } y = \pm\infty, \tag{3.3}$$

where, for the mixing layer, it is easily shown (following GN05) that

$$\begin{aligned}\mathcal{N} \equiv (\Phi' - c)(\mathrm{D}^2 - \alpha^2) - \Phi''' + \frac{\mathrm{i}}{\alpha Re}\Big\{ &\mathrm{D}^4 - 2\alpha^2\mathrm{D}^2 + \alpha^4 + p\Phi\mathrm{D}^3 \\ &+ p(\Phi'\mathrm{D}^2 + \Phi''\mathrm{D}) - \alpha^2[2py(\Phi' - c) + p\Phi]\mathrm{D} - p\alpha^2 c + p\Phi''' \\ &+ (3\Phi' - c)Re\,\alpha\alpha' + [\Phi''' + \alpha^2(3\Phi' - 2c) - \Phi'\mathrm{D}^2]Re\frac{\partial}{\partial x}\Big\}.\end{aligned}$$

Here D is the derivative with respect to y, $c \equiv \omega/\alpha$ is the (in general complex) phase speed of the disturbance, the prime on α denotes a derivative with respect to x, $Re \equiv 2\Lambda U_\infty \delta_\omega/(1 + \Lambda)\nu$ and $p \equiv Re(\mathrm{d}\delta_\omega/\mathrm{d}x_d) = \Lambda(\delta_\omega/l_d)/(1 + \Lambda)$. Note that in the minimal composite theory the non-parallel flow operator $\mathcal{N}$ is flow-specific, and so is different e.g. from that for the boundary layer.

Following GN97 $\mathscr{N}$ is now expressed as the sum of an operator $\mathscr{M}$ that contains all the lowest-order terms and an operator $\mathscr{H}$ comprising the higher-order terms,

$$\left.\begin{aligned}\mathscr{M} &\equiv (\Phi' - c)(\mathrm{D}^2 - \alpha^2) - \Phi''' + \frac{\mathrm{i}}{\alpha Re}\mathrm{D}^4,\\ \mathscr{H} &\equiv \mathscr{N} - \mathscr{M}\\ &= \frac{\mathrm{i}}{\alpha Re}\Big\{ p\Phi\mathrm{D}^3 - 2\alpha^2\mathrm{D}^2 + \alpha^4 + p(\Phi'\mathrm{D}^2 + \Phi''\mathrm{D}) - \alpha^2[2py(\Phi' - c) + p\Phi]\mathrm{D}\\ &\quad - p\alpha^2 c + p\Phi''' + (3\Phi' - c)Re\alpha\alpha' + Re\mathscr{S}\frac{\partial}{\partial x}\Big\},\end{aligned}\right\} \tag{3.4}$$

where

$$\mathscr{S} \equiv [\Phi''' + \alpha^2(3\Phi' - 2c) - \Phi'\mathrm{D}^2].$$

In making this decomposition it must be noted that the relative order of magnitude of each individual term varies with y because of the presence of the critical layer at $\Phi' = c$. In constructing the minimal composite equation that yields eigenfunctions correct up to $O(Re^{-1})$, the order of magnitude of any term within the y-domain is considered. The final equation consists of all terms that are at least $O(Re^{-1})$ somewhere, and omits all that are $o(Re^{-1})$ everywhere in the domain. It must be emphasized that the above equations are different from those in GN05 for a boundary layer. For instance, in the construction of the operator $\mathscr{M}$ for the present case clearly no wall-layer considerations are relevant. Furthermore, the critical point is found to be located close to the inflection point, their separation in y being always less than two-thirds of the critical layer thickness $(\alpha R)^{-1/3}$. We can therefore take y_c to be small. It follows that the term $p\Phi\mathrm{D}^3$ is of higher order everywhere in the mixing layer, so can be conveniently included in the operator $\mathscr{H}$, whereas in the boundary layer the term is generally part of $\mathscr{M}$. The solution procedure for estimating the growth of the disturbance, though, remains the same, and is given in GN05 in detail. For reference, we recapitulate some of its essential points below.

While the non-parallel operator $\mathscr{N}$ has partial derivatives in both x and y, the lowest-order terms comprising $\mathscr{M}$ contain derivatives only in y. The total solution ϕ can therefore be conveniently expressed as the sum

$$\phi(x, y) = A(x)\phi_m(x, y) + \epsilon\phi_h(x, y), \tag{3.5}$$

where ϕ_m is the lowest-order eigenfunction which satisfies the equation

$$\mathscr{M}\{\phi_m\} = 0, \tag{3.6}$$

and $A(x)$ is its amplitude as a function of x. In non-parallel flow theory, the marginal stability boundary varies depending on the position y, the disturbance quantity selected and the trajectory along which it is monitored. One parameter for which the curve of marginal stability depends only on x is the integral of the disturbance kinetic energy, given by

$$\left.\begin{aligned}&K_d = \left(\frac{2\Lambda U_\infty}{1+\Lambda}\right)^2 \frac{|A|^2}{4}\exp\left(-2\int \alpha_i\,\mathrm{d}x\right)\int_{-\infty}^{\infty}\mathrm{d}y\left(\frac{|u|^2 + |v|^2}{2}\right),\\ &\text{where}\quad u = \frac{\partial\phi_m}{\partial y},\qquad v = -\left(\frac{p}{Re} + \mathrm{i}\alpha + \frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}x}\right)\phi_m - \frac{\partial\phi_m}{\partial x} + \frac{py}{Re}\frac{\partial\phi_m}{\partial y}\end{aligned}\right\} \tag{3.7}$$

and α_i is the imaginary part of the complex wavenumber α. The downstream growth of the disturbance kinetic energy is therefore

$$g \equiv \frac{1}{K_d}\frac{\mathrm{d}K_d}{\mathrm{d}x}. \tag{3.8}$$

The derivatives of ϕ_m and α with respect to x, used in (3.4) and (3.7) for instance, are obtained by solving (3.6) at a nearby Re for an identical dimensional frequency ω_d. This procedure also yields K_d for two nearby Reynolds numbers. Thus the growth rate g given by (3.8) is known, subject to determining the quantity $\mathrm{d}(\log A)/\mathrm{d}x$, which can be done as follows.

Substituting from (3.5) into (3.3), and noting that $\mathscr{N} = \mathscr{M} + \mathscr{H}$, we obtain the amplitude evolution equation

$$A\mathscr{H}\phi_m + \frac{\mathrm{d}A}{\mathrm{d}x}\mathscr{S}\phi_m = -\epsilon\mathscr{M}\phi_h, \tag{3.9}$$

the truncated terms here being $O(Re^{-2/3})$ compared to the largest of the retained terms.

The expression for the adjoint of the operator $\mathscr{M}$ is found to be

$$\overline{\mathscr{M}} = (\Phi' - c^*)(\mathrm{D}^2 - \alpha^{*2}) + 2\Phi''\mathrm{D} - \frac{\mathrm{i}}{\alpha^* Re}\mathrm{D}^4 + O\left(Re^{-2/3}\right),$$

where a quantity superscripted with an asterisk represents its complex conjugate. Using the property of adjoints, the contribution to the growth g due to $\mathrm{d}(\log A)/\mathrm{d}x$ can be calculated without the need to specifically compute the higher-order solution ϕ_h (cf. GN05). Equation (3.9) therefore reduces to

$$A\int_{-\infty}^{\infty}\chi^*\mathscr{H}\{\phi_m\}\,\mathrm{d}y + \frac{\mathrm{d}A}{\mathrm{d}x}\int_{-\infty}^{\infty}\chi^*\mathscr{S}\{\phi_m\}\,\mathrm{d}y = \text{(higher-order terms)},$$

where χ is the solution of the adjoint problem

$$\overline{\mathscr{M}}\{\chi\} = 0.$$

Now $A \sim O(1)$ and $\mathrm{d}A/\mathrm{d}x$ is $O(Re^{-1})$, so the error in the estimation of the growth rate and therefore the boundary of marginal stability will be $o(Re^{-1})$. But as we integrate over a large streamwise distance of $O(Re)$, the error in the amplitude is expected to be $o(1)$.

Note that, in minimal composite theory, an approximate solution of the partial differential equation (3.3) is obtained, to a certain order of accuracy, by solving an ordinary differential equation and using the property of adjoints; the higher-order terms $\mathscr{H}$ are required to calculate the amplitude function $A(x)$, but not the eigenfunction ϕ_m. An obvious advantage is that the complexity of the problem is significantly reduced when compared to the exercise of solving it through the full partial differential equations.

4. Results

Both parallel and non-parallel stability analyses involve solving an eigenvalue problem. The conditions at infinity are posed at distances sufficiently far away to ensure that results are independent of the size of the domain. For example, on the curve of marginal stability, the domain is chosen in such a way that any further increase in its size does not affect the Reynolds number to within 10^{-4} for a given α. We notice that at large y, Φ''' vanishes and the higher derivatives of the eigenfunction with respect to

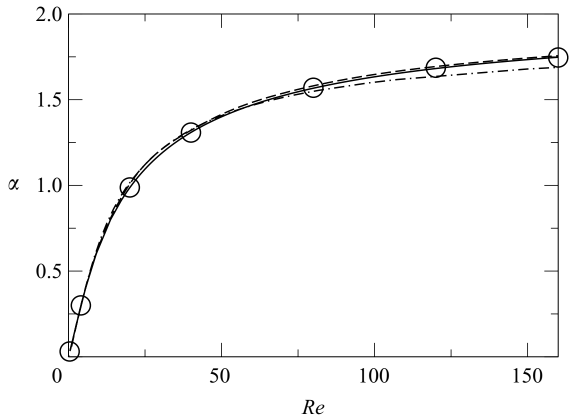

FIGURE 2. Results of OS (parallel) analysis: circles, Betchov & Szewczyk (1963); solid line, present analysis on a tanh-profile; dashes, similarity profile with $\Lambda = 1$; dash-dot line, similarity profile with $\Lambda = 1/3$.

y (appearing in the viscous term) become smaller. Therefore, the eigenfunction decays like $\exp(\pm\alpha y)$ as $y \to \pm\infty$, as in the inviscid solutions. The y-domain thus needs to be larger for smaller wavenumbers. Further, since the eigenfunctions are discretized as eigenvectors in the numerical solution, a sufficient number of grid points must be contained in the domain so that the results are independent of resolution.

For the range of non-dimensional wavenumbers considered, i.e. from 0.2 to 2.0, the domain specifications vary. At the higher end of the wavenumber range, outer boundary conditions are imposed at $y = \pm 4$ with an 81 grid-point spread, whereas for lower wavenumbers the domain is bounded at $y = \pm 60$ and has 401 grid points. A sinh grid stretch is used to give a higher resolution near the critical layer. On the lower limit of the stability loop obtaining numerically accurate results becomes computationally expensive at higher Reynolds numbers; hence the calculations do not go below a non-dimensional wavenumber of 0.2. The marginal stability boundary is defined as where the streamwise growth rate $|g|$ for neutrally stable temporal modes (ω real) is smaller than 1×10^{-5} in magnitude. In calculating the streamwise derivatives, nearby stations are taken corresponding to an increment of 1 % in the Reynolds number Re. Independence of the curve of marginal stability from small deviations in the numerical values of these parameters has been confirmed. Two independent developments of the computer code give the same results.

First, as one test of how sensitively results depend on the precise velocity profile, we have compared the results of Betchov & Szewczyk (1963) with those of the present analysis for the hyperbolic tangent profile as well as the similarity profiles for $\Lambda = 1, 1/3$ (figure 2). The match is found to be excellent at all wavenumbers. For the tanh-profile the scaling is chosen to make the vorticity thickness identical to that of the similarity solution for the corresponding velocity ratio. In all cases $Re_{cr} = 0$, showing that the Betchov–Szweczyk result is not specific to a counterflow situation, but instead is inherent to the parallel flow assumption.

Next, the results of the parallel and non-parallel analyses are compared in figure 3 for the four Λ-values 1, 3/5, 1/3 and 1/7. (Incidentally these values of Λ correspond to the slower moving stream introduced at the inlet at a uniform speed being 0, 1/4, 1/2 and 3/4 times that of the faster stream respectively.) The results correspond to the respective similarity profiles. It may be expected that the difference between the

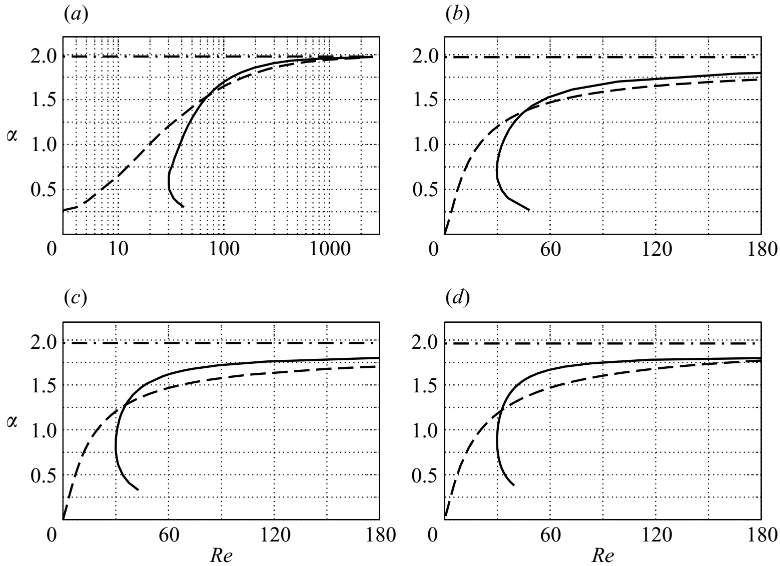

FIGURE 3. Comparison of the parallel and non-parallel analyses: Reynolds number vs. the real part of the wavenumber α. Dashed line, OS; solid line, non-parallel; the horizontal dash-dotted lines show the solution of the Rayleigh equation for each velocity ratio. (*a*) $\Lambda = 1$, (*b*) 3/5, (*c*) 1/3, (*d*) 1/7.

Λ	1/39	1/7	1/3	3/5	1
Re_{cr}	29.8	29.8	29.9	29.7	30.1

TABLE 1. The critical Reynolds numbers for different velocity ratios.

parallel and non-parallel flow results diminishes in the limit $Re \to \infty$. Figure 3 shows that the curves of marginal stability from the two analyses do indeed approach the marginally stable mode of the solution to Rayleigh's equation for each of the velocity ratios considered above. Note that the curves for $\Lambda = 1$ have been plotted with Re on a log scale to show that the results are indistinguishable as $Re \to \infty$.

In non-parallel theory a disturbance may in general amplify at one y value and decay at another, and one disturbance quantity could amplify while others decay. The curve of marginal stability and hence the critical Reynolds number can therefore vary, as already mentioned, depending on the quantity whose growth rate is sought and the trajectory in the (x, y)-plane along which it is determined. The monitoring location makes a great difference to the stability results in the case of boundary layers (GN97) but this was not found to be the case in the present flow. Over the range $y = -0.2$ to $y = 0.1$ the critical Re for the streamwise disturbance velocity u was observed to vary within 10 % of its value at $y = 0$. We do not expect any widely different behaviour outside this y region as the variation in the velocity profile there becomes negligible. It is also interesting to note that within the range of accuracy of the present calculations the minimum value of this critical Reynolds number for any given velocity ratio occurs at the inflection point in the streamwise velocity profile.

The critical Reynolds number Re_{cr} for the disturbance energy K_d is shown in table 1 for different velocity ratios. Within the range of accuracy of the present calculations,

Λ	h	Re_m
1/7	0.0001197	46.8
1/3	0.0002678	48.7
3/5	0.0003067	40.8
1	0.0003969	32.3

TABLE 2. Norm of the $\mathscr{H}$ operator. Re_m is the value of the Reynolds number at which h takes the maximum value along the marginal stability curve for the given Λ.

this value remains nearly constant at about 30, even as Λ goes to 0. Drawing any conclusions about the small deviations from this number would not be justified. A Reynolds number based on conventional scales, $U_\infty l_d/\nu$, varies on the other hand from 6 at $\Lambda = 1$ to ∞ at $\Lambda = 0$: the latter result shows that in the limit of shearless flow the flow is stable at all Reynolds numbers based on U_∞, l_d.

5. Error analysis

As the present analysis is (to our knowledge) the first to provide a non-zero critical Reynolds number for a mixing layer, there are no more accurate results to compare with. An assessment of the accuracy of the present results can therefore proceed along two lines.

First, we can construct a measure of the error in the present analysis in relation to a solution of the full non-parallel equation (3.3). As the chief approximation made here is the neglect of the higher-order operator $\mathscr{H}$, an obvious measure of the error is

$$h \equiv \frac{\langle \mathscr{H}\phi_m, \mathscr{H}\phi_m \rangle}{\langle \phi_m, \phi_m \rangle}, \tag{5.1}$$

where angular brackets denote the inner product

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f g^* \, \mathrm{d}y.$$

The quantity h is related to the norm of the operator $\mathscr{H}$, which is defined here as the smallest number h for which $|\mathscr{H}\phi_m\| \leqslant h|\phi_m\|$ (following Liusternik & Sobolev 1961, p. 82), evaluating the right-hand side of (5.1) along the marginal stability loop.

The values of h for different values of Λ are listed in table 2. The highest value, which occurs at $\Lambda = 1$, is of order 4×10^{-4}; it drops to 1×10^{-4} at $\Lambda = 1/7$. This suggests that the error is indeed small, particularly at small Λ.

The second approach is to examine estimates of the error in the other stability calculations using minimal composite theory (MCT). GN05 report that, on a Blasius boundary layer, the difference in the logarithm of the amplitude ratio calculated by MCT and the direct linearized Navier–Stokes solutions computed by Fazel & Konzelmann (1990), is less than 3% at a Reynolds number of 375 based on free-stream velocity and momentum thickness. The errors may be expected to increase as the Reynolds number falls. In a recent stability calculation of the flow through a divergent channel, where the profile involves an inflection point, it has been found that the errors are somewhat larger, being of order 15% at a critical Reynolds number of 77 as calculated by full non-parallel theory (K. Sahu 2006, private communication).

To obtain a better assessment one would need to do a full linear DNS study or a series of experiments, both of which are outside the scope of the present paper.

6. Conclusions

The main result of the present work is the demonstration of the existence of a non-zero critical Reynolds number for a laminar plane incompressible mixing layer. Choosing vorticity thickness and velocity differential as length and velocity scales, this critical Reynolds number is about 30 for all values of Λ between 0 to 1.

We found that for all the parallel and inviscid analyses carried out on the similarity profiles, the results differ from corresponding analyses on the (suitably rescaled) tanh-profiles by no more than 5 %. Furthermore, Orr–Sommerfeld analysis (assuming parallel flow) yields $Re_{cr} = 0$ whether the velocity profile is tanh or a similarity solution. It follows that the marked deviation observed in the final result of the present work must therefore be attributed to flow non-parallelism, and not to differences in assumed mean velocity profiles.

The present analysis is therefore a striking example of a flow where the use of non-parallel theory is essential to avoid obtaining non-physical results. Though parallel flow theory has given revealing insights into instability mechanisms, there are thus flow regimes where it can be qualitatively wrong.

The finding of a critical Reynolds number which is not too low is consistent with the assumptions of our non-parallel analysis. Flow non-parallelism can be measured through the quantity $(\alpha_d \delta_\omega)^{-1}\, \mathrm{d}\delta_\omega/\mathrm{d}x_d$ which is equal to p/Re for non-dimensional wavenumbers of $O(1)$. For the smallest Reynolds number at which instability is found, this quantity diminishes almost linearly from about 1/12 at $\Lambda = 1$ to zero at $\Lambda = 0$. The laminar similarity flow analysed here is physically realizable only some distance downstream of the splitter plate. Therefore the critical Reynolds number found here also has the appealing consequence that the results may be tested experimentally or by direct numerical simulation. We note that since the flow is convectively stable below $Re = 30$, the question of absolute instability below this Reynolds number does not arise. While it is outside the thrust of the present work, it is relevant to mention that from an inviscid parallel flow analysis, Huerre & Monkewitz (1985) have shown that the instability is convective for any mixing layer formed between co-flowing streams.

Of course the present approach is a modal theory, and it does not preclude significant transient growth in disturbance amplitude even at Reynolds numbers lower than the critical value calculated here.

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